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A one-dimensional electron system in the Luther–Emery regime

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Abstract. The one-dimensional (1D) electron gas model in the strong coupling regime is considered. It is shown that this model is Bethe ansatz solvable and the exact spectrum of the Hamiltonian is obtained. The massive spin elementary excitations are established based on the Bethe ansatz equations. The magnetization near the onset at zero temperature and the low-temperature thermodynamics are calculated.

1. Bethe ansatz equations

1D electron systems have been studied extensively in the past three decades and many interesting exact results had been obtained both via bosonization techniques [1–3] and the Bethe ansatz [4, 5]. Another important method to approach 1D systems is the renormalization group theory which, with the so-called 'g-ology', groups the 1D electron systems into a few universal classes [6, 7]. The g-ology includes a few constants (functions) to describe the interactions among the electrons. g_4 and g_2 describe the forward scattering processes of the electrons with the same moving direction and different moving directions, respectively; g_1 describes the backward scattering across the two Fermi points and g_3 the $4k_F$ (Umklapp) process. In the unhalf-filled systems, the $4k_F$ process is highly oscillatory and is often omitted. g_4 only induces the renormalization of the Fermi velocity or other parameters [6]. It does not affect the fixed point physics of the system and can also be omitted. Usually, g_i's are also labelled with $\|$ and $\perp$ to describe the spin-parallel and spin-opposite scattering processes.

In this paper, we consider the 1D electron interacting model in the strong coupling regime. This problem was first considered by Luther and Emery via bosonization. Exact results were obtained in their paper [3] for a very special model. In this case, the backward scattering becomes relevant and it opens a gap in the spin excitation spectrum. The system, in the language of renormalization group theory, will never flow to the Luttinger liquid fixed point. The Hamiltonian we shall consider reads

$$\begin{aligned} H = \int \Bigg\{ & -\mathrm{i} \sum_{\beta=\pm 1} \sum_{s=\pm 1} \beta c^{\dagger}_{\beta s}(x) \frac{\partial}{\partial x} c_{\beta s}(x) + g_2 \sum_{ss'} c^{\dagger}_{1s} c^{\dagger}_{-1s'} c_{-1s'} c_{1s} \\ & - g_{1\|} \sum_{s} c^{\dagger}_{1s} c_{1s} c^{\dagger}_{-1s} c_{-1s} + g_{1\perp} \sum_{s} c^{\dagger}_{1s} c^{\dagger}_{-1-s} c_{-1s} c_{1-s} \Bigg\} \mathrm{d}x \end{aligned} \tag{1}$$

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where $\beta = \pm 1$ denotes the chiral indices of the electrons and $s = \pm 1$ denotes the spin components of the electrons; $c^{\dagger}_{\beta s}$ ($c_{\beta s}$) is the creation (annihilation) operator of electrons. For simplicity, we have put the Fermi velocity $v_{\rm F} = 1$. This model is just the anisotropic case of the chiral Gross–Neveu model which was solved by Andrei and Lowenstein [8] and independently by Belavin [9]. The present model is also Bethe ansatz soluble.

The exact solution for the eigenstates and eigenvalues of Hamiltonian (1) can be obtained within the framework of the Bethe ansatz method [10]. The central object of this method is the two-particle scattering matrix S which is calculated from the two-particle processes described by the Hamiltonian (1):

$$S_{ij} = \begin{cases} \exp\{\frac{1}{4}{\rm i}(\alpha_i - \alpha_j)[2g_2 - g_{1\parallel} - g_{1\parallel}\sigma_i^z\sigma_j^z + g_{1\perp}(\sigma_i^x\sigma_j^x + \sigma_i^y\sigma_j^y)]\} & \text{for } \alpha_i \neq \alpha_j \\ p_{ij} & \text{for } \alpha_i = \alpha_j \end{cases} \tag{2}$$

where $\alpha_{i,j} = \pm 1$ are the chiralities of the momenta and σ is the Pauli matrix. p_{ij} is the spin exchange operator. The system is integrable as the two-particle scattering matrix (2) satisfies the Yang–Baxter equation [4, 11]. Furthermore, there are no genuine N-particle scattering processes for $N \geqslant 3$ due to the δ-function form of the interactions in (1). Therefore the mathematical conditions for eigenstates of (1) in the form of the Bethe ansatz are satisfied. For details we refer the reader to the literature. The S-matrix (2) can be factorized as follows:

$$S_{ij} = {\rm e}^{{\rm i}(\alpha_i - \alpha_j)\phi}[\omega_0 + \omega_z\sigma_i^z\sigma_j^z + \omega_\perp(\sigma_i^x\sigma_j^x + \sigma_i^y\sigma_j^y)] \tag{3}$$

with

$$\phi = \tfrac{1}{4}{\rm i}(2g_2 - 2g_{1\parallel} - g_{1\perp}) \qquad \omega_0 - \omega_z = \frac{\sin[(\alpha_i - \alpha_j)f]}{\sin[(\alpha_i - \alpha_j)f + {\rm i}\mu]}$$

$$2\omega_\perp = \frac{{\rm i}\sinh\mu}{\sin[(\alpha_{\rm i} - \alpha_j)f + {\rm i}\mu]} \qquad \omega_0 + \omega_z = 1 \tag{4}$$

and

$$\cosh\mu = \frac{\cos g_{1\parallel}}{\cos g_{1\perp}} \qquad \cot^2 2f = \frac{\sin^2 g_{1\parallel}}{\sin(g_{1\perp} - g_{1\parallel})\sin(g_{1\parallel} + g_{1\perp})}. \tag{5}$$

Above we have put $|g_{1\parallel}| < |g_{1\perp}| \leqslant \pi/2$. In this case, the condition $f/\mu > 0$ is satisfied. For finite systems, suitable boundary conditions have to be imposed, e.g. periodic ones. In this case, the momentum k_j of each electron is subject to certain quantization conditions involving the total scattering phase upon other particles. This leads to the diagonalization problem of products of scattering matrices solved by a subsequent Bethe ansatz [4]. The result is given in terms of spin rapidities λ_α and the nested Bethe ansatz equations

$$\begin{aligned} {\rm e}^{{\rm i}k_j L} &= {\rm e}^{2{\rm i}N_-\phi}\prod_{\gamma=1}^{M}\frac{\sin[\lambda_\gamma - f + \frac{1}{2}{\rm i}\mu]}{\sin[\lambda_\gamma - f - \frac{1}{2}{\rm i}\mu]} \\ {\rm e}^{{\rm i}q_l L} &= {\rm e}^{-2{\rm i}N_+\phi}\prod_{\gamma=1}^{M}\frac{\sin[\lambda_\gamma + f + \frac{1}{2}{\rm i}\mu]}{\sin[\lambda_\gamma + f - \frac{1}{2}{\rm i}\mu]} \\ \left\{\frac{\sin[\lambda_\gamma - f + \frac{1}{2}{\rm i}\mu]}{\sin[\lambda_\gamma - f - \frac{1}{2}{\rm i}\mu]}\right\}^{N_+} & \left\{\frac{\sin[\lambda_\gamma + f + \frac{1}{2}{\rm i}\mu]}{\sin[\lambda_\gamma + f - \frac{1}{2}{\rm i}\mu]}\right\}^{N_-} = -\prod_{\delta=1}^{M}\frac{\sin[\lambda_\gamma - \lambda_\delta + {\rm i}\mu]}{\sin[\lambda_\gamma - \lambda_\delta - {\rm i}\mu]} \end{aligned} \tag{6}$$

where the k's (q's) are the momenta carried by the right (left) going electrons and the λ's are the rapidities of spins. N_+ (N_-) is the number of right (left) going electrons. L denotes the length of the system. Generally we put $N_+ = N_- = N$, $N/L = D$. The eigenenergy defined by $\{k_j, q_l, \lambda_\gamma\}$ is

$$E = \sum_j \frac{2\pi}{L} n_j^+ - \sum_l \frac{2\pi}{L} n_l^- + D \sum_\gamma [\Theta(\Lambda_\gamma - f) - \Theta(\lambda_\gamma + f) - 2\pi + 2\mu] \tag{7}$$

where $n_j^\pm$ are integers or half odd integers denoting the charge quanta; $\Theta(x) = -2\arctan(\coth(\mu/2)\tan x)$. Note the irrelevant constant $2(N_+ + N_-)\phi$ has been omitted in equation (7) and we have taken the Landre factor of the electrons as $1 - \mu/\pi$ for the anisotropy [10].

2. Ground state and elementary excitations

For the ground state, all $n_j^\pm$'s are consecutive numbers and all λ's are real. In the thermodynamic limit $L \to \infty$, $N_+/L = N_-/L \to D$, from equation (6) we deduce that the density of λ, e.g. $\sigma_0(\lambda)$, satisfies the following integral equation

$$N[a_1(\lambda - f) + a_1(\lambda + f)] = \sigma_0(\lambda) + \int_{-\pi}^{\pi} a_2(\lambda - \lambda')\sigma_0(\lambda')\,\mathrm{d}\lambda' \tag{8}$$

with

$$a_n(\lambda) = \frac{1}{\pi}\frac{\sinh(n\mu)}{\cosh(n\mu) - \cos(2\lambda)}. \tag{9}$$

We denote the Fourier transformation of a function $g(\lambda)$ in the interval $(-\pi, \pi]$ as

$$\tilde{g}(m) = \int_{-\pi}^{\pi} g(m)\mathrm{e}^{\mathrm{i}m\lambda}\,\mathrm{d}\lambda \qquad m = \text{integer}. \tag{10}$$

From equation (8) we have

$$\tilde{\sigma_0}(m) = \frac{N\cos(mf)}{\cosh(m\mu/2)} \tag{11}$$

which gives the total spin of the ground state as

$$S = N - \tilde{\sigma_0}(0) = 0 \tag{12}$$

and the density of λ for the ground state

$$\sigma_0(\lambda) = \frac{N}{2\pi}\sum_{m=-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}m\lambda}\frac{\cos(mf)}{\cosh(m\mu/2)}. \tag{13}$$

Equation (13) can be expressed by the Jacobian elliptic function $\mathrm{dn}(x)$.

The more physically interesting objects are the elementary excitations. From equation (7) we see that the charge and spin excitations can be separately described while the charge subsector behaves as a non-interacting spinless fermion system and the only non-trivial excitations are the spin ones. There are two types of spin excitations. One is the spin triplet which can be obtained by putting two holes in the λ sea of the ground state. The other is the spin singlet which can be described by a 2-string of λ and two holes. First, we consider the spin triplet state. In this case, the density of λ satisfies the following integral equation:

$$N[a_1(\lambda - f) + a_1(\lambda + f)] = \sigma(\lambda) + \sum_{i=1}^{2}\delta(\lambda - \lambda_i^h) + \int_{-\pi}^{\pi} a_2(\lambda - \lambda')\sigma(\lambda')\,\mathrm{d}\lambda' \tag{14}$$

with λ_i^h the position of the ith hole. Through Fourier transformation we have

$$\delta\tilde{\sigma}(m) = \tilde{\sigma}(m) - \tilde{\sigma_0}(m) = -\frac{e^{im\lambda_1^h} + e^{im\lambda_2^h}}{1 + \exp(im\mu)}. \tag{15}$$

The excitation energy ΔE is then given by

$$\Delta E = \frac{D}{2\pi} \sum_{m=-\infty}^{\infty} \int_{-\pi}^{\pi} d\lambda e^{-im\lambda} \frac{e^{im\lambda_1^h} + e^{im\lambda_2^h}}{1 + \exp(im\mu)} [2\pi - 2\mu - \Theta(\lambda - f) + \Theta(\lambda + f)]. \tag{16}$$

It is very hard to calculate the infinite summation in equation (16) directly. However, in the present model we can take the scaling limit $D \to \infty$, $\mu \to 0$, $\pi\lambda_i^h/\mu \to \chi_i^h$ and keep $T_k = De^{-\pi f/\mu}$ finite as in the Kondo model [12]. It has been demonstrated in [12] that, at least to the leading term, such a scaling scheme coincides with the conventional lore. In this case, we can replace the summation $\sum_m$ by $\frac{1}{\mu}\int d(\mu m)$. The final result is

$$\Delta E = 4T_k \cosh \chi_1^h + 4T_k \cosh \chi_2^h. \tag{17}$$

The above procedure can be applied to the multi-hole case. The excitation energy is just the summation of those of single holes. Also, the spin singlet excitation can be obtained by the same procedure. In this case, $\sigma(\lambda)$ satisfies

$$N[a_1(\lambda - f) + a_1(\lambda + f)] = \sigma(\lambda) + \sum_{i=1}^{2} \delta(\lambda - \lambda_i^h) + \int_{-\pi}^{\pi} [\sigma(\lambda') + \sigma_s(\lambda')] a_2(\lambda - \lambda')\, d\lambda' \tag{18}$$

where

$$\sigma_s(\lambda) = \delta(\lambda - \bar{\lambda} + i\mu/2) + \delta(\lambda - \bar{\lambda} - i\mu/2) \tag{19}$$

is the density of the 2-string and $\bar{\lambda} = (\lambda_1^h + \lambda_2^h)/2$ is the centre of the string. We do not repeat the calculation here. The excitation energy has the same form of equation (17). That means that the direct contribution of the 2-string is exactly cancelled by the back flow of the λ-sea induced by it. Even so, it does not mean that the two types of excitations are degenerate. In fact, we can calculate the scattering matrices of the massive particles and find that they are different for different types of excitation [12].

3. Zero temperature magnetization

We proceed by considering the magnetization of the system at zero temperature. When the system is subjected to an external magnetic field H, the magnetic moments of the electrons couple to this field. Because of the spin gap in the spin excitation spectrum, there must be a critical field H_c to flip the spins in the ground state. As we know, a single λ-hole is only a spin-$\frac{1}{2}$ object [13]. A spin flip must correspond to two λ-holes. Then we have the critical field as

$$H_c = 8T_k. \tag{20}$$

When H exceeds the critical value, λ-holes around zero may be generated to compensate the magnetic energy gain. No string presents at zero temperature because it contributes higher magnetic energy. The density of λ now satisfies

$$\sigma_B(\lambda) = N[a_1(\lambda - f) + a_1(\lambda + f)] - \int_{-\pi}^{\pi} a_2(\lambda - \lambda')\sigma_B(\lambda')\theta(|\lambda'| - B)\, d\lambda' \tag{21}$$

where $\theta(x)$ is the step function. The spin-dependent part of the energy is

$$E(S) = D\int_{-\pi}^{\pi}[\sigma_B(\lambda)\theta(|\lambda| - B) - \sigma_0(\lambda)][\Theta(\lambda - f) - \Theta(\lambda + f) - 2\pi + 2\mu]\,\mathrm{d}\lambda$$
$$-(1 - \frac{\mu}{\pi})HS \qquad (22)$$

and the total spin (magnetization) is given by

$$S = \frac{1}{2}\int_{-B}^{B}\sigma_B(\lambda)\,\mathrm{d}\lambda. \qquad (23)$$

Through the Fourier transformation we rewrite equation (21) and equation (22) as

$$\sigma_B(\lambda) = \sigma_0(\lambda) + \int_{-B}^{B} R(\lambda - \lambda')\sigma_B(\lambda')\,\mathrm{d}\lambda' \qquad (24)$$

$$E(S) = \int_{-B}^{B} 4T_k \cosh\frac{\pi}{\mu}\lambda\sigma_B(\lambda)\,\mathrm{d}\lambda - (1 - \frac{\mu}{\pi})HS \qquad (25)$$

with

$$R(\lambda) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}m\lambda}\frac{\sin((\pi - 2\mu)m/2)}{\sin(\pi m/2) + \sin((\pi - 2\mu)m/2)}. \qquad (26)$$

For small $B \ll 1$, near the onset of the magnetization, we can expand equation (23) by

$$S = \sigma_B(0)B + \tfrac{1}{6}\sigma_B''(0)B^3 + \mathrm{O}(B^5). \qquad (27)$$

We use the same procedure to iterate equation (24) and then substitute $\sigma_B(\lambda)$ into equation (25). After this we minimize $E(S)$ with respect to S to lead to

$$S/L = \frac{H_\mathrm{c}}{\sqrt{2}\pi}\left(\frac{H - H_\mathrm{c}}{H_\mathrm{c}}\right)^{\frac{1}{2}}\left[1 + \mathrm{O}\left(\frac{H - H_\mathrm{c}}{H_\mathrm{c}}\right)\right]. \qquad (28)$$

4. Thermodynamics

The thermodynamics of the present model can be constructed by the standard method [14–16]. At finite temperature, the λ's may take complex values which are grouped in various strings. An n-string is characterized by a common real abscissa $\lambda_{n,\gamma}$ and an order n:

$$\lambda_{n,\gamma}^{j} = \lambda_{n,\gamma} + \mathrm{i}\frac{\mu}{2}(n + 1 - 2j) \qquad j = 1, 2, \ldots, n. \qquad (29)$$

Define the density of n-strings and the density of n-string holes as $\sigma_n(\lambda)$ and $\sigma_n^h(\lambda)$, respectively. They satisfy the following integral equation:

$$N[a_n(\lambda - f) + a_n(\lambda + f)] = \sigma_n^h(\lambda) + \sum_{l=1}^{\infty} A_{nl}\sigma_l(\lambda) \qquad (30)$$

$$A_{nl} = [|l - n|] + 2[|l - n| + 2] + \cdots + 2[l + n - 2] + [l + n] \qquad (31)$$

where $[n]$ is an integral operator with the kernel $a_n(\lambda)$. After some manipulations [12] we get the integral equations for $\eta_n(\lambda) = \sigma_n^h(\lambda)/\sigma_n(\lambda)$ as

$$\ln\eta_n = G[\ln(1 + \eta_{n+1}) + \ln(1 + \eta_{n-1})]$$
$$\ln\eta_1 = -\frac{4T_k}{T}\cosh(\pi\lambda/\mu) + G\ln(1 + \eta_2)$$
$$\lim_{n\to\infty}\{[n + 1]\ln(1 + \eta_n) - [n]\ln(1 + \eta_{n+1})\} = -\frac{2H}{T} \qquad (32)$$

where G is an integral operator with the kernel

$$g(\lambda) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}m\lambda}}{2\cosh(\mu m/2)}. \tag{33}$$

The free energy is given by

$$F = E_0 - \frac{\pi L T^2}{6} - T \int_{-\pi}^{\pi} \mathrm{d}\lambda\, \sigma_0(\lambda) \ln[1 + \eta_1(\lambda)] \tag{34}$$

with $\sigma_0(\lambda)$ the density of λ for the ground state and E_0 a constant. Change the variable to $\chi = (\pi\lambda/\mu)$ and take the scaling limit $\mu \to 0$. We have

$$F = E_0 - \frac{\pi L T^2}{6} - \frac{2T_k T L}{\pi} \int_{-\infty}^{\infty} \cosh\chi \ln[1 + \eta_1(\chi)]\, \mathrm{d}\chi. \tag{35}$$

There have been a few studies in the literature [15, 16] on the asymptotic solution of equation (32). We shall not specify it here. The final results in some special parameter regions are summarized as follows:

4.1. $T \ll H, T \ll H_\mathrm{c} - H$

In this region, the electrons are strongly coupled and most of the physical quantities have an activation law via temperature. The associated magnetization, susceptibility per unit length and specific heat are

$$S = \frac{L H_\mathrm{c}}{4\sqrt{\pi}} \left(\frac{T}{H_\mathrm{c}}\right)^{\frac{1}{2}} \exp\left[-\frac{H_\mathrm{c} - H}{2T}\right] \left[1 + \frac{3T}{4H_\mathrm{c}} + \mathrm{O}\left(\frac{T^2}{H_\mathrm{c}^2}\right)\right] \tag{36}$$

$$\chi = \frac{1}{8\sqrt{\pi}} \left(\frac{H_\mathrm{c}}{T}\right)^{\frac{1}{2}} \exp\left[-\frac{H_\mathrm{c} - H}{2T}\right] \left[1 + \frac{3T}{4H_\mathrm{c}} + \mathrm{O}\left(\frac{T^2}{H_\mathrm{c}^2}\right)\right] \tag{37}$$

$$\begin{aligned} C = {}& \frac{\pi}{3} T + \frac{H_\mathrm{c}}{8\sqrt{\pi}} \left(\frac{T}{H_\mathrm{c}}\right)^{\frac{1}{2}} \left(\frac{H_\mathrm{c} - H}{T}\right)^2 \\ & \times \exp\left[-\frac{H_\mathrm{c} - H}{2T}\right] \left[1 + \frac{3T}{4H_\mathrm{c}} + \frac{2T}{H_\mathrm{c} - H} + \mathrm{O}\left(\frac{T^2}{H_\mathrm{c}^2}\right)\right]. \end{aligned} \tag{38}$$

The square root law via temperature in the above equations cannot be deduced from the perturbation theory.

4.2. $T \ll H, |H_\mathrm{c} - H| \ll T \ll H_\mathrm{c}$

In this region, the susceptibility and the specific heat per unit length are given by

$$\chi = \frac{1}{4\sqrt{\pi}} \left(\frac{H_\mathrm{c}}{T}\right)^{\frac{1}{2}} \left[\frac{1}{2}\eta\left(-\frac{1}{2}\right) + \frac{3}{8}\eta\left(\frac{1}{2}\right)\frac{T}{H_\mathrm{c}} + \mathrm{O}\left(\frac{T^2}{H_\mathrm{c}^2}, \frac{H - H_\mathrm{c}}{T}\right)\right] \tag{39}$$

$$\begin{aligned} C = {}& \frac{\pi}{3} T + \frac{3H_\mathrm{c}}{8\sqrt{\pi}} \left(\frac{T}{H_\mathrm{c}}\right)^{\frac{1}{2}} \left[\eta\left(\frac{3}{2}\right) + \frac{15}{4}\eta\left(\frac{5}{2}\right)\frac{T}{H_\mathrm{c}} - \frac{1}{6}\eta\left(-\frac{1}{2}\right)\frac{H - H_\mathrm{c}}{T} \right. \\ & \left. + \mathrm{O}\left(\frac{T^2}{H_\mathrm{c}^2}, \frac{(H - H_\mathrm{c})^2}{T^2}, \frac{H - H_\mathrm{c}}{H_\mathrm{c}}\right)\right] \end{aligned} \tag{40}$$

with $\eta(x) = (1 - 2^{1-x})\zeta(x)$, where $\zeta(x)$ is the Riemann zeta function. Note that when $H = H_\mathrm{c}$, there is a critical behaviour $\chi \sim T^{-\frac{1}{2}}$, $C \sim T^{\frac{1}{2}}$ as $T \to 0$.

4.3. $T \ll H$, $H_c \ll H$

In this region, the strong magnetic field destroys the spin gap seriously. Then the electrons fall into the weak coupling regime. The susceptibility and the specific heat have the following asymptotic forms:

$$\chi(H,T) = \chi(H,0) + \frac{\alpha\pi T^2}{24H^2}\left\{\ln^{-3}(\alpha H/H_c) + \mathrm{O}\left[\frac{\ln\ln(\alpha H/H_c)}{\ln^4(\alpha H/H_c)}\right]\right\} \qquad (41)$$

$$C(H,T) = \frac{\pi}{3}T\left\{2 + \frac{\alpha}{8\ln^2(\alpha H/H_c)} + \mathrm{O}\left[\frac{\ln\ln(\alpha H/H_c)}{\ln^3(\alpha H/H_c)}\right]\right\} \qquad (42)$$

where $\alpha = 2(2e/\pi)^{\frac{1}{2}}$ is a constant. It is clearly shown that the specific heat recovers the linear dependence on the temperature.

5. Concluding remarks

In conclusion, we have established the exact solutions of the 1D electron model in the strong coupling regime. The present model is just the anisotropic version of the backward scattering model. Based on the Bethe ansatz equations, the spin excitations, the onset magnetization at zero temperature and the thermodynamics for some parameter regions are calculated. The results for the present model coincide very well with the predictions of the renormalization group theory [6].

References

[1] Mattis D C and Lieb E H 1965 *J. Math. Phys.* **6** 304
[2] Tomonaga S 1950 *Prog. Theor. Phys.* **5** 554
[3] Luther A and Emery V J 1974 *Phys. Rev. Lett.* **33** 589
[4] Yang C N 1967 *Phys. Rev. Lett.* **19** 1312
[5] Lieb E H and Wu F Y 1968 *Phys. Rev. Lett.* **20** 1445
[6] Sólyom J 1979 *Adv. Phys.* **28** 201;
for a recent review, see Voit J 1995 *Rep. Prog. Phys.* **58** 977
[7] Emery V J 1979 *Highly Conducting One-Dimensional Solids* (New York: Plenum)
[8] Andrei N and Lowenstein J H 1979 *Phys. Rev. Lett.* **43** 1698
[9] Belavin A A 1979 *Phys. Lett.* **87B** 117
[10] Tsvelick A M and Wiegmann P B 1983 *Adv. Phys.* **32** 453
[11] Baxter R J 1982 *Exactly Solved Models in Statistical Physics* (London: Academic)
[12] Andrei N, Furuya K and Lowenstein J H 1983 *Rev. Mod. Phys.* **55** 331
[13] Faddeev L D and Takhtajan L A 1981 *Phys. Lett.* **85A** 375
[14] Gaudin M 1971 *Phys. Rev. Lett.* **26** 1301
[15] Japaridze G I and Nersesyan A A 1983 *Phys. Lett.* **94A** 224
[16] Bahder T B, Rezayi E H and Sak J 1985 *J. Phys. C: Solid State Phys.* **18** 899